How central agents affect group behaviour and the speed of learning

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Abstract

Centralized network structures, which consist of at least one actor – a *central agent* – whose behaviour is observed by the rest of the group are predominant across social, political and economic settings. In public and corporate sectors, central agents such as managers and CEOs, shape organizational culture and identity, which in turn affects organizational performance. This paper studies how central agents affect behaviour formation through social learning. We show that although central agents play a crucial role in driving the group to a consensus, they do not necessarily exert the most influence on the group's equilibrium behaviour. In equilibrium, an agent's influence corresponds to her eigenvector centrality. We also examine the convergence rate of behaviour formation and show that it depends on the degree to which central agents facilitate group cohesion.

Keywords: Networks, central agents, behaviour formation, social learning.

JEL: D83, D85

1. Introduction

Centralized network structures are observed in many social, economic and political settings. What distinguishes centralized networks from other structures is that they consist of at least one agent – a *central agent* – whose actions and opinions are observed by the rest of the group. The media play the role of central agents by transmitting information to mass audiences and supplying information that people use to make various decisions, ranging from voting to participation in public programs.[1] In corporate and public sectors, managers and executive directors act as central agents whose behaviour is observed by the rest of the employees. Research in organizational science demonstrates that managers directly influence the development of an

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[1] This phenomenon is evident in social media (e.g. Twitter and Facebook), which provides alternative communication platforms that allows member of the media and others in the society to become central agents (Golbeck et al., 2010; Chadwick and Stromer-Galley, 2016; Borge Bravo and Esteve Del Valle, 2017).

organization's culture and identity, which form through the process of social learning.[2] An organization's culture and identity in turn affects its performance.[3] Despite their political and economic relevance, the literature lacks a framework for examining how central agents influence group behaviour formation.

Can central agents drive the group to achieve a consensus (i.e. to have identical opinions and behaviour), and if so, how fast can the outcome be achieved? How much influence do central agents exert on group behaviour? Answers to these questions will shed light on how to design policies aimed at regulating group behaviour in centralized networks. Consider an example of a government interested in promoting a policy or public program. The relevant questions may range from whether mass media can be used to drive the society to hold a shared positive opinion towards a policy, and how fast it can be achieved. In addition, who holds the largest influence on overall group opinion, mass media or some other members of the society? In a similar vein, managers may be interested to know how much influence they hold within the organization and which relationships to target in order to influence organizational culture and identity.

To develop answers to above questions, we draw insights from the literature of social learning. We specifically assume that people adjust their behaviour to conform with the actions and opinion of those with whom they share close ties. The empirical evidence of social conformism dates back to the work of Asch and Guetzkow (1951) on social pressure. More recently, Falk and Ichino (2006), Zafar (2011) and Abeler et al. (2011) provide evidence of social conformism in work habits and efforts; and Kahan et al. (2012) find evidence of conformism in opinion formation. Depending on the context, close ties can represent family, friendship and mass media-audience relationships, or work-related relationships such as belonging to the same department, office and who-reports-to-whom relations. Close ties are assumed to be heterogeneous and non-reciprocal so that people attach different weights to others' actions and opinions. The weights individuals attach to each other's behaviour represents the extent to which they value each other's actions and opinions and may depict the level of trust or the frequency of interaction between them. Under this structure, central agents are those whose behaviour receives positive weights from the rest of the group. Using this framework, we examine: (*i*) the

[2]See for example Bass (1985), Bass and Avolio (1993), Scott and Lane (2000) and Boehm et al. (2015) who show that managers play a role in the development of an organizations culture; and Besharov (2014) who shows how organizational identity arise from social learning between members of the organization.

[3]Barney (1986), Gordon and DiTomaso (1992), Marcoulides and Heck (1993) and Voss et al. (2006) provide empirical evidence showing that an organization's culture and identity affects its performance and success.

nature and composition of equilibrium behaviour in centralized networks, and (*ii*) the speed of convergence of the behaviour formation process.

We proceed by first examining how central agents affect equilibrium behaviour. We show that the presence of at least one central agent is sufficient to drive the group to a consensus. This implies that within an organization, if every member attaches a positive weight to the behaviour of managers, no matter how small, then all members of the organization will hold identical work habits and views about the values that should drive the organization. Similarly, in public sectors, all connected departments will hold identical work and ethical habits (e.g. the extent to which individuals engage in corruption).

Although central agents play a crucial role in driving the group to a consensus, they are not necessarily the most influential in shaping group behaviour. We show that in equilibrium, the extent of an agent's influence corresponds to her *eigenvector centrality*. An agent's eigenvector centrality increases with the total weight that other agents attach to her behaviour, and with the eigenvector centralities of her neighbours. For example, in political networks, the central agents, i.e. mass media and politicians, are not necessarily the most influential in shaping society's attitudes towards policies and public programs. Other members of the society such as activists and lobby groups may be the most influential.

If central agents drive a society to a consensus, why is heterogeneity in behaviour observable in many centralized social and organizational structures? The answer to this question lies in the speed of convergence to a consensus, which we find to depend on the extent to which central agents hold the group together. We derive a measure of group cohesion for centralized networks as the sum of the "weakest relationships" that central agents hold with other members of the group. If the group is highly cohesive, then it converges rapidly to a consensus; otherwise disagreement persists for longer periods. Our results provide insights into which relationships to target in order to rapidly achieve a consensus in centralized networks.

In addition to the papers discussed above, our paper contributes to the literature on social learning, which cuts across fields. It includes DeGroot (1974), DeMarzo et al. (2003), Golub and Jackson (2010), Acemoğlu et al. (2013) and Melguizo (2016) in economics; DeGroot (1974), Friedkin and Johnsen (1990) and Friedkin et al. (2016) in sociology; and Hegselmann et al. (2002), and Olfati-Saber et al. (2007) in computer science. A recurring theme in this literature is that a consensus emerges in the long-run if a network is strongly connected and does not consist of cycles, otherwise disagreement persists in equilibrium.[4] Disagreement also persists in

[4] A group of agents is said to be strongly connected in W if for every pair in the group, there exists a path

the presence of stubborn agents who do not place any weight on others' behaviour (Acemoğlu et al., 2013). We significantly improve on this literature by showing that the sufficient condition for a consensus to emerge, regardless of whether the network is strongly connected or cycles exist, is the existence of at least one central agent.[5]

The remainder of this paper is organized as follows. Section 2 formulates the model of behaviour formation through social learning. Section 3 characterizes equilibrium behaviour of the model, showing that a consensus emerges in centralized network. Section 4 studies the speed of learning. The conclusion is offered in Section 5 and proofs are relegated to the Appendix.

2. A model of behaviour formation through social learning

2.1. Interaction structure

We consider a group of agents of finite size denoted by the set $N = \{1, 2, \cdots, n\}$. The composition of the group is allowed to be heterogeneous. For example, in political settings, it can consist of mass media, politician and ordinary citizens; in corporate sectors, it may consists of managers, CEOs and employees. Within N, each agent observes the behaviours of only a subset of other agents. For each agent i, we denote by N_i the subset of agents whose behaviour i observes, and refer to it as the *neighbourhood* of i.

We use network theory to represent the interaction structure. In particular, we represent individual interactions by an $n \times n$ non-negative interaction matrix, here denoted by W. For each pair of agents i and j, an element w_{ij} of W indicates the weight that i attaches to j's behaviour. These weights represent the value agents attach to each other's actions or opinions, and may capture the level of trust or the frequency of interaction between them. We consider a general case where interactions are directed so that $w_{ij} > 0$ need not imply $w_{ji} > 0$, or equality between w_{ij} and w_{ji}. For example, a political figure can disseminate her opinions regarding an issue to her social media followers and not attach any importance to the opinions displayed in their replies. The interaction matrix W is normalized to be row-stochastic; that is, its rows sum to one. This normalization is customary in the literature and avoids situation where some agents' influence grows without bounds in the dynamic process to be described shortly.

of influence from one agent to the other. If cycles exist, then agents within the cycle will alternate in adopting each other's actions and the process never settles into a consensus.

[5]In Section 4, we discuss how our findings on the speed of learning relate to Golub and Jackson (2012) who study a similar concept in homophilous networks.

2.2. Examples of centralized networks

A central agents are those whose behaviour is observed by every other agent in the society. Formally, this means that if i is central, then $w_{ij} > 0$ for all $j \in N$. Below, we present examples of centralized networks that are representative of real-world social, economic and political networks.

Figure 1a is representative of political networks that can be partitioned into subgroups based on political orientation. In Figure 1a, the society consists of two subgroups: the left leaning $g_1 = \{1, , 2, 3, 4, 5\}$ and the right leaning $g_2 = \{6, , 7, 8, 9, 10\}$. Associated to these subgroups are two media sources or political figures A and B, where A is left leaning and B is right leaning. All agents listen and attach positive weight to opinions of both media sources but the weights that members in g_1 attach to B is less than the weights they attach to A, and vice versa for members of subgroup g_2. Members within subgroups interact locally and interactions across subgroups need not occur but if they do, then inter-group weights are much less than intra-group weights. The two media sources may also attach positive weights to some members of the society, e.g. politicians, lobby groups and so on. For this type of networks, our model provides insights on how media sources or political figures influence equilibrium opinions of the society and the speed of learning.

Figure 1b represent a hierarchical network that is characteristic of corporate organizational structures. It represents situations where a society, in this case a corporation, is structured into multiple hierarchies. Members within each hierarchy interact with each other and across hierarchies, and all agents observe the behaviour of those at the top of the hierarchy. For this family of networks, our model helps explain how central agents (e.g. manager) may influence work habits and attitudes of other members of the organization, which in turn affects organizationals performance.

Figure 1c is a network where the society is divided into sub-communities that all interact with central actors. These networks are representative of structures found in many public sectors, where multiple departments interact with the central supervisory or managing department. In this case our model helps explain how the network structure influences ethical behaviours like corruption. The extent to which an actor engages in corrupt activities depends on the willingness of other member within and across departments to also engage in corruption.

2.3. Payoffs and dynamics: social conformism

We consider a behaviour formation process where agents repeatedly minimize the cost of miscoordinating their behaviour with that of their neighbours. We specifically consider a coordi-

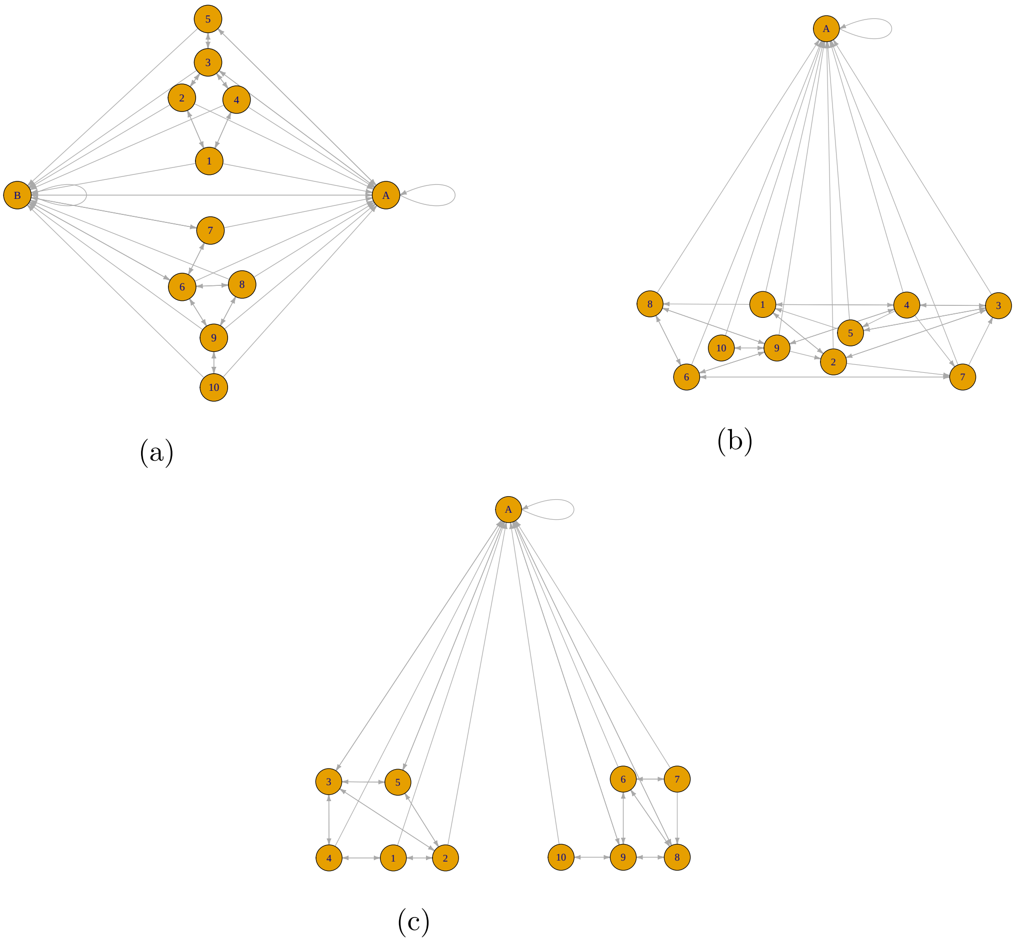


Figure 1: Examples of centralized networks that are representative of real-world networks: (a) is representative of political networks, (b) is representative of organizational networks, and (c) is representative of networks found in public sectors.

nation problem where individuals desire to conform their behaviour to that of their neighbours. Social conformism is prevalent in a wide range of social, political and economic settings. Social conformism has been observed in work habits and contributive games (Carpenter, 2004; Falk and Ichino, 2006; Zafar, 2011; Abeler et al., 2011). There is also evidence from social psychology research showing that attitude conflicts in relationships are a source of psychological stress and instability (Heider, 1946; Festinger, 1962). More recently, Kahan et al. (2012) observe that individuals conform to beliefs of those with whom they share close ties.

The phenomenon of conformism can be captured in the simplest form by assuming that agents incur a disutility from misaligning their actions with those of others with whom they share close ties. Formally, let $a_i \in [0, 1]$ be the action of agent i, represented by a positive real number in the interval from zero to one. Here, a_i can represent an agent's choice of political involvement (e.g. level of engagement in political actions such as protests), or level of effort to exert in organizational duties. It could also represent a unidimensional attitude toward others' attributes, group behaviour, or government policies and public programs; see for example Poole and Daniels (1985) and Ansolabehere et al. (2008) who provide empirical evidence showing that individual opinions on a wide range of issues spanning domains such as politics, lifestyles and the economy, can be described using a unidimensional spectrum. The choice of bounds fo a_i does not influence our results in a qualitative sense and assuming a continuous interval suitably describes actions where agents choose levels of engagement, efforts or contribution, and the case of unidimensional opinions.

Let $\mathbf{a}$ be a configuration vector representing actions taken by all agents. We assume that agents simultaneously minimize the disutility of miscoordination at discrete time interval $t = 1, 2, \cdots$. More specifically, let $a_i(t)$ and $\mathbf{a}(t)$ be the respective action and vector of actions at t. Then each agent i incurs a disutility $U_i(a_i(t), \mathbf{a}(t-1))$ for misaligning actions $a_i(t)$ at time t, with the observed actions of her neighbours at time $t-1$, where

$$U_i(a(t), \mathbf{a}(t-1)) = -\sum_{j \in N} w_{ij}(a_j(t-1) - a_i(t))^2 \tag{1}$$

The action that minimizes (1) is then

$$a_i(t) = \sum_{j \in N} w_{ij} a_j(t-1) \quad \text{for } i = 1, 2, \cdot, n. \tag{2}$$

Given the interaction matrix W and noting that $\mathbf{a}(0)$ represents a vector of initial actions, the system of equations in (2) can be expressed in matrix form as

$$\mathbf{a}(t) = W\mathbf{a}(t-1) \tag{3}$$

The dynamic system (3) describes evolution of actions of a group of agents interacting through a social network with interaction matrix W. In the case of behaviour formation, the initial vector $\mathbf{a}(0)$ can be interpreted as the actions that individuals would take in the absence of social influence. For opinion formation process, the initial vector corresponds to prior beliefs. In both case, $\mathbf{a}(0)$ is determined by agents' environment, and their prior experiences, such as education, political and religious orientation, and so on. From the second equality of (3), we see that if agents start with the same action, it stays so forever.[6] This is a characteristic of behaviour and opinion formation processes with pure coordination externalities. It also indicates that a consensus is an equilibrium, although it need not be in cases where agents start with heterogeneous actions.

[6]That is, if $a_i = a$ for all $i \in N$, then $\mathbf{a}(0) = a\mathbf{e}$, where $\mathbf{e}$ is a vector of ones. Since W^t is row-stochastic, $\mathbf{a}(t) = W^t\mathbf{a}(0) = aW^t\mathbf{e} = a\mathbf{e}$. And hence, even after t iterations, all agents still choose action a.

Our subsequent analysis considers situations where the initial vector is not homogeneous. We characterize equilibrium behaviour, focusing on the role that central agents play. Specifically, we show in the next section that the presence of central agents is sufficient to drive the group to a consensus regardless of the distribution of initial actions. We then examine whether the strategic position of central agents directly translates into greater influence on group behaviour. In Section 4, we examine how central agents affect the speed of convergence of an evolutionary process described by (3). We develop a network based measure that quantifies the extent to which central agents influence the speed of evolution.

3. How central agents influence equilibrium behaviour

This section characterizes equilibrium of a behaviour formation process represented by (3). We first show that the presence of at least one central agent is sufficient to drive the group to a consensus. We then distinguish between the neighbourhood size and influence, and show that a large neighbourhood size need not directly translate into greater influence on overall group behaviour.

3.1. Equilibrium behaviour: central agents and consensus

In general networks, the dynamics in (3) converges to either a consensus, where equilibrium actions are identical, or to equilibrium disagreement where agents behave differently. Let $\mathbf{a}^* = \lim_{t\to\infty} \mathbf{a}(t)$ be a vector of equilibrium actions. If (3) converges to a consensus, then $a_i^* = a_j^*$ for all pairs of agents i and j. The definition equilibrium consensus leads to the following lemma.

Lemma 1. *Let* $\mathbf{e}$ *be a column vector of ones. Then a consensus occurs only if there exists a non-zero vector* π *so that* $\lim_{t\to\infty} W^t = \mathbf{e}\pi^T$.

Proof. See Appendix A.1 □

The matrix $\mathbf{e}\pi^T$ consists of identical rows of π, and in equilibrium, agent i's action is

$$a_i^* = \sum_{j=1}^{n} \pi_j a_j(0) \tag{4}$$

Hence, if the society converges to a consensus, then π describes the level of influence that agents exert on each others' equilibrium actions. That is, the behaviour and opinions of agents with the largest influences, as measured by π_i', are the most adopted in equilibrium. For each i, π_i depends on i's position in the network. In the following proposition, we show that the presence of a central agent is sufficient to generate a consensus in equilibrium.

Proposition 1. *If a society consists of at least one agent for whom $w_{ij} > 0$ for all $j \in N$, then there exists a unique well-defined vector π so that $\lim_{t\to\infty} W^t = \mathbf{e}\pi^T$.*

Proof. See Appendix A.2

□

Proposition 1 states that the presence of at least one central agent in the group is sufficient to generate a consensus. Central agents thus act as "mediators" of group behaviour. For the case of public sector networks described above, Proposition 1 implies that in equilibrium, all sub-departments will adopt a similar behaviour; that is, all departments will exert the same level of effort, and exercise similar levels of corruption.

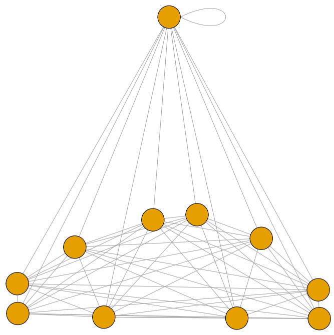
The equilibrium behaviour that the group converges to is determined by the composition of the *influences vector* π. The influence vector is equivalent to the left eigenvector of the interactions matrix corresponding to the leading eigenvalue $\lambda_1 = 1$; as such, π_i is sometimes referred to as the *eigenvector centrality* of agent i (Bonacich, 1987). We can thus derive π as a solution to the set of simultaneous equations $\pi^T = \pi^T W$, which in turn implies that[7]

$$\pi_i = \sum_{j=1}^{n} w_{ji}\pi_j \tag{5}$$

From (5) we see that the eigenvector centrality accords each agent a level of influence that depends on her total direct influence as measured by the sum $w_i = \sum_{j=1}^{n} w_{ji}$, and on the *quality* of her neighbours. The quantity w_i is the total weight that all other agents attach to i's behaviour. The quality of i's neighbours corresponds to the eigenvector centralities π_j of all j that attach positive weight to i's behaviour. If other agents attach sufficiently large weights to a central agent, then the latter will have the largest eigenvector centrality, and hence the largest influence on group behaviour.

Example 1: Consider a hierarchical network in Figure 2 that is representative of organizational structures as discussed above. As depicted by the interaction matrix on the right hand side of Figure 2, this network has one central agent at the top hierarchy. All other agents at the lower hierarchy interact with the central agent and with each other. The level of influence that other agents exert on the central agent varies with the value of the parameter $0 \leq \epsilon \leq 1$. When $\epsilon = 0$, the central agent is not influenced by those at the lower hierarchy and hence acts as an enforcer of behaviour and rules. As ϵ tends to one, the central agent's influence diminishes and

[7] Let $\Pi = \mathbf{e}\pi^T$. Since Π is derived by infinitely iterating W, then $\Pi^t = \Pi$ and $\Pi = \Pi W$. This implies that for each influence vector π, $\pi^T = \pi^T W$, and hence π is a left eigenvector of W corresponding to the leading eigenvalue $\lambda_1 = 1$.



$$W = \begin{bmatrix} 1-\epsilon & \frac{\epsilon}{n-1} & \frac{\epsilon}{n-1} & \cdots & \frac{\epsilon}{n-1} \\ \frac{1}{n-1} & 0 & \frac{1}{n-1} & \cdots & \frac{1}{n-1} \\ \frac{1}{n-1} & \frac{1}{n-1} & 0 & \cdots & \frac{1}{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{n-1} & \frac{1}{n-1} & \frac{1}{n-1} & \cdots & 0 \end{bmatrix}$$

Figure 2: An example of a hierarchical network, with the respective interaction matrix W on the right hand side.

the network tends to a horizontal structure where all agents hold the same level of influence. For this network structure, the eigenvector centralities are: $\pi_1 = \frac{1}{1+(n-1)\epsilon}$ for the central agent, and $\pi_i = \frac{\epsilon}{1+(n-1)\epsilon}$ for all other agents $i \geq 2$.[8] We see that when $\epsilon = 0$, the central agent commands complete influence and all other agents conform to her behaviour in equilibrium. In the example of work habits discussed above, this implies that the entire organization conforms to the levels of effort exerted by the agents at the top of the hierarchy.

From a theoretical point of view, Proposition 1 significantly improves existing results on convergence and consensus of the learning dynamics represented by (3). The existing results show that a consensus arises if and only if the society is made up of exactly one *strongly connected* and *closed group* of agents and that W is *aperiodic* on that group (Jackson, 2008, Chapter 8, Corollaries 8.1 & 8.2). A group of agents is said to be strongly connected in W if for every pair in the group, there exists a path of influence from one agent to the other. That is, for any pair of agents i and j, either i is directly influenced by j, or there exists a set of agents $i = i_1, \cdots, i_k, \cdots, i_K = j$ whereby there is a sequence of links $i \to i_2 \to \cdots \to i_{K-1} \to j$ from i to j, and vice versa. If a strongly connected group is *closed*, then it is not influenced by other agents outside of the group. If a society consists of more than one closed group, then disagreement can persist in the long-run. Indeed, Acemoğlu et al. (2013) show that the presence of "stubborn" agents, that is agents who are not influenced by others, leads to disagreement in equilibrium opinions. Aperiodicity on the other hand ensures that no cycles exist within a strongly connected and closed subgroup. For example, if a cycle exists among a group of agents $\{i, j, k\}$, then the interactions are of the form $i \to j \to k \to i$. In such situations, agents within a cycle alternate in adopting each others actions and the process never settles into a consensus.

Proposition 1 states that what matters for a consensus is a presence of a central agent,

[8] See (Desai and Rao, 1993) for the derivations.

regardless of whether the network is strongly connected or cycles exists within subgroups of the society. Existence of central agents rules out the possibility of stubborn agents. This is because if there are at least two central agents, then by definition they directly influence each other, even if only weakly.

3.2. Neighbourhood size vs influence

In Example 1 above, the central agent is able to command the largest influence for all values of $\epsilon \leq 1$. This is because agents at the bottom hierarchy interact uniformly and hence attach sufficiently large weights to the central agent's behaviour. Overall, however, it is possible for some agents at the lower hierarchy to have eigenvector centralities that are greater than for the central agent. Being centrally placed thus need not directly translate into greater influence on group behaviour.

Recall that N_i denotes the neighbourhood of i excluding i. Let $N_j(i) = N_j \cap N_i$ be the set of neighbours shared by i and j; and let $w_j(i) = \sum_{k \in N_j(i)} w_{kj}$ be the total weight that agents in $N_j(i)$ attach to j. Note that if j is a central agent, then $N_j(i) = N_i$ plus i. The following proposition shows that having a large neighbourhood size need not translate to greater influence on group behaviour.

Proposition 2. *Let j be a central agent, which by definition means $w_{kj} > 0$ for all $k \in N$. For any other i who is not a central agent, $w_i > w_j(i)$ can be chosen so that $\pi_i > \pi_j$.*

Proof. See Appendix A.3

□

Proposition 2 shows that it is possible for an agent who is not central to outrank a central agent in terms of eigenvector centrality, provided the former has loyal followers who attach large weights to her behaviour. This result underscores the complexity of adaptive processes with externalities. Results from several empirical studies that use online social networks are consistent with the predictions of our model. In online social networks such as "Twitter", the number of "retweets" is a good measure of how ones ideas and opinions have diffused through the network, and hence a measures of a user's influence. The neighbourhood size of an actor in the network on the other hand corresponds to the number of her twitter followers. Several studies have empirically examined the relationship between neighbourhood size and influence using twitter networks.

Cha et al. (2010) analyse a large data set generated from twitter network and "...conclude that the most connected users are not necessarily the most influential when it comes to engaging one's audience in conversations and having one's messages spread." They arrive at this conclusion by examining the relationship between neighbourhood sizes of agents, as measured by the number of followers, and the number of retweets that the user generates. They find a very weak correlation between neighbourhood size and retweets. The empirical findings by Cha et al. (2010) are consistent with the findings in Kwak et al. (2010) who analyse the entire twitter network, Karlsen and Enjolras (2016) who examine twitter usage by Norwegian politicians, and Vaccari and Valeriani (2015) who examine social media usage by Italian politicians; they all find that larger neighbourhood size does not directly translate into greater influence.

Although the analysis in above papers applies to the diffusion of a single opinion contained in a single tweet, the observed empirical discrepancy between neighbourhood size and influence is consistent with the rationale underlying the results of Proposition 2. The empirical results suggests that the extent to which an opinion contained in a tweet diffuses through retweets depends on the total weight that agents attach to each other's opinions. The total weights correspond to the quantity w_i for each i. In a similar vein, Proposition 2 shows that what matter for the level of influence that the agent commands in the group is w_i. In a more complex process where each agent shares her opinions with others in network through retweets, equilibrium influences are determined by eigenvector centralities. The eigenvector centralities are collectively determined by w_is.

4. The speed of learning: central agents and group cohesion

In this section, we examine how central agents affect the speed of learning. Since the presence of central agents ensures a consensus in equilibrium, the speed of learning is equivalent to the speed of reaching a consensus. Studying the speed of learning is relevant because in some cases achieving a consensus may take many iterations, and updating may be infrequent. Under such scenarios, a policy maker or leaders of an organization can identify which links to target in order to rapidly attain a consensus.

We define the speed of learning as the time it takes the dynamic process (3) to get close to equilibrium. To formalize this notion, we first define what it means to be close to equilibrium. Recall that in equilibrium, $a_i^* = a^* = \sum_{j=1}^{n} \pi_j a_j(0)$, identical for all agents. Equilibrium action a^* is a weighted average of initial or prior actions, where the weighting vector π consists of individual influences in equilibrium. Each $\pi_j a_j(0)$ is then the contribution of agent j to the average action in equilibrium. We can compute the average action at time t as $\bar{a}(t) = \sum_{j=1}^{n} \pi_j(t) a_j(0)$, where $\pi_j(t)$ is the overall influence of j after t iterations. Since the contribution of each agent to average action evolves over time, we then define the distance $DC(t; W; \mathbf{a}(0))$

to equilibrium after t iterations under W and for initial vector $\mathbf{a}(0)$ as

$$DC(t; W; \mathbf{a}(0)) = \sum_{i=1}^{n} |\pi_i(t)a_i(0) - \pi_i a_i(0)| \tag{6}$$

where $|\pi_i(t)a_i(0) - \pi_i a_i(0)|$ is the distance of i's contribution to average action at t, to her contribution to equilibrium average action. As the process evolves to equilibrium, the distance $DC(t; W; \mathbf{a}(0))$ decays to zero. The speed of learning is then the time it takes this distance to get sufficiently close to zero. That is, for some small real number $\varepsilon > 0$, the convergence time, or consensus time, $CT(\varepsilon; W)$ is the time it takes $DC(t; W; \mathbf{a}(0))$ to get below ε.

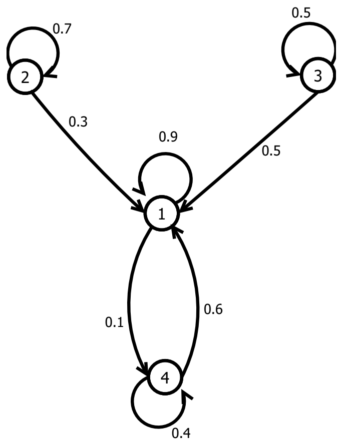
Definition 1. *The consensus time $CT(\varepsilon; W)$ to $\varepsilon > 0$ under interaction matrix W is*

$$CT(\varepsilon; W) = \sup_{\mathbf{a}(0)\in\mathbb{R}^n} \min\{t : \; DC(t; W; \mathbf{a}(0)) < \varepsilon\} \tag{7}$$

It is clear from the definition in (6) that the choice of an initial vector greatly determines the distance to equilibrium at any t. The initial vector can be chosen so that the distance to equilibrium is close to zero after a few iterations. The supremum on the right hand side of (7) ensures that we consider the worst possible choice of the initial vector. Doing so also enables us to focus on the effect of the network structure on convergence time and not the choice of an initial vector.

Our main result in this section establishes the extent to which central agents influence the convergence time. We show that it is the *weakest interactions* that greatly influence the speed of learning and not the neighbourhood size of central agents or density of the network. By weakest interactions we mean the smallest weights that other agents attach to the central agents' behaviours. The intuition behind this relationship is that a complete consensus is reached in centralized societies only once the agents who are least influenced by the central agent agree with the central agent's actions and opinions. To see how, consider an example of a centralized network depicted in Figure 3 with the respective interaction matrix on the right hand side. The eigenvector centralities to the sixth decimal place are $\pi = (0.857143, 0, 0, 0.142857)$. By iterating W, we find that agents 1 and 4 converge to their respective centralities of 0.857143 and 0.142857 after 11 steps of updating. Agent 3 converges to her centrality of zero after 18 steps of updating, and agent 2 after 37 steps. Clearly, the weakest link, from agent 2 to the central agent, determines when complete consensus occurs.

To formalize this argument, let $\mathbf{q}$ be a nonnegative column vector derived from W as follows. Define $q_i \equiv \min_{1\leq j\leq n} w_{ji}$ for $i = 1, 2, \cdots, n$, so that $\mathbf{q} = (q_1, q_2, \cdots, q_n)^T$. Each q_i is the least intensity of attention that i receives from other agents; $q_i > 0$ only if all agents pay attention



$$W = \begin{bmatrix} 0.9 & 0 & 0 & 0.1 \\ 0.3 & 0.7 & 0 & 0 \\ 0.5 & 0 & 0.5 & 0 \\ 0.6 & 0 & 0 & 0.4 \end{bmatrix}$$

Figure 3: An example of a centralized network, with the respective interaction matrix W on the right hand side.

to i including self-influence, it is zero otherwise. The vector $\mathbf{q}$ is thus only non-zero if and only if there is at least one central agent. Given W and $\mathbf{q}$, we then define a parameter $\beta = \sum_{i \in N} q_i$.

The parameter β measures *network cohesion*, or the extent to which central agents bring the society together. This is because central agents act as "social connectors" by reducing the path lengths between any pair of agents. Any pair of agents who are not directly connected influence each other's behaviour through the central agent. If all agents attach higher weights to central agents' behaviours, then the society is highly cohesive with very short distances between any pair of agents; consequently β will be large. If on the other hand there are agents who attach small weights to central agents', then their connections act as the "weakest bridges" or "bottlenecks" between the section of the group that interacts with these agents and central agents. The following proposition establishes the relationship between consensus time and parameter β.

Proposition 3. *The convergence time in the model of learning described by (3) is bounded by*

$$\frac{\ln(\varepsilon/2r)}{\ln\left(1-\beta\right)} - 1 \leq CT(\varepsilon; W) \leq \frac{\ln(\varepsilon/2)}{\ln\left(1-\beta\right)} - 1 \tag{8}$$

where $0 \leq r \leq \frac{1}{\beta}\left(\sum_{i=1}^{n} q_i^2\right)^{\frac{1}{2}} \leq 1$.

Proof. See Appendix A.4

Proposition 3 shows that the convergence time decreases with β. If central agents do not have the capacity to keep the society cohesive so that β is very small, then disagreement in actions may persist over extended periods. Centralization of the group is thus not sufficient to rapidly drive it to a consensus. Instead, it is the extent to which they facilitate group cohesive.

Consider an example of a political network of Figure 1a, with two central agents A and B representing left and right leaning political media sources respectively. From the definition of $\mathbf{q}$ and β, it follows that $\beta = q_A + q_B$, where $q_A \equiv \min_{1\leq j\leq n} w_{jA}$ and $q_B \equiv \min_{1\leq j\leq n} w_{jB}$ are respectively the least weights that a left-leaning agent attaches to a right-leaning media source, and the least weight that a right-leaning agent attaches to a left leaning media source. The smaller these two values, the smaller is β, and the longer disagreement persists in the society.

Our results have policy implications for governments and organizations that aim to reduce disagreement. Our results suggest that the crucial relationships for targeting are the weakest ties. In most social, economic and political settings, the relationships with the lowest trust are usually apparent, and hence, it is not costly to identify them. The costly aspect comes in at the stage of providing suitable incentives to strengthen such relationships. The decision that governments and organizations face in such settings is to weigh the costs of providing incentives to strengthen the weakest ties, against the benefits of achieving a consensus.

Our results are related to Golub and Jackson (2012) who show that the speed of naïve learning decreases with the second largest eigenvalue of the network. Golub and Jackson (2012) also show that the second largest eigenvalue measures the extent of homophily in the network, which is the tendency of individuals to attach most weight to the behaviour of those with whom they share attributes e.g. race, political orientation, income classs, e.t.c. This interpretation directly implies that overall network cohesion should be proportional to one minus the second largest eigenvalue of the interaction matrix. The following corollary establishes the relationship between the second largest eigenvalue and β.

Corollary 1. *Let W be an interaction matrix with the second largest eigenvalue $\mu_2(W)$, and let β be as defined above. Then $\mu_2(W) \leq 1 - \beta$*

Proof. See Appendix A.5

Corollary 1 shows that the results of Proposition 3 not only establish the role of central agents in influencing the speed of learning, they also provide an alternative characterization. The computation and interpretation of the second largest eigenvalue requires complete knowledge of the network structure. The weakest relationships to the central agents on the other hand are easier to identify and quantify and have straightforward interpretations.

5. Concluding remarks

The question of how peoples' attitudes and behaviours evolve is at the centre of behavioural sciences. The models of adaptive learning with naïve rules have been instrumental in explaining the processes of opinion formation and behavioural change. Empirical studies also suggests that people indeed tend to follow naïve rules of learning. The objective of this paper has been to examine how central agents influence the process of behaviour formation through social learning. Our study is motivated by the observation that centralized organizational structures are predominant across social, economic and political settings. The advent of internet and social media has also made it possible for many public figures to become central agents, in addition to the traditional central agents such as radio and TV broadcasters.

We have shown that the presence of at least one central agent is sufficient to drive the society to a consensus. In equilibrium, the level of influence that central agents exert on others' behaviour is determined by how much weight that other agents attach to the formers' actions. Central agents are not necessarily the most influential however. We also examined how central agents affect the speed of convergence to a consensus. We showed that in societies with at least one central agent, the speed of learning depends on the weakest links to the central agents. The direct implication is that centralization is not a sufficient to rapidly drive the society to a consensus. Our results have direct policy implications for organizations and governments that aim to reduce disagreement.

Appendix A. Appendix

Appendix A.1. Proof of Lemma 1

First note that (3) can be rewritten as $\mathbf{a}(t) = W^t\mathbf{a}(0)$ so that the vector of equilibrium actions $\mathbf{a}^*$ is given by

$$\mathbf{a}^* = \lim_{t\to\infty} W^t\mathbf{a}(0) \tag{A.1}$$

From the right hand side of (A.1), if $\lim_{t\to\infty} W^t = \mathbf{e}\pi^T$, then $\mathbf{a}^* = \mathbf{e}\pi^T\mathbf{a}(0)$, which implies that $a_i^* = \sum_{j=1}^n \pi_j a_j(0) = \sum_{k=1}^n \pi_k a_k(0) = a_j^*$ for all i and j.

Appendix A.2. Proof of Proposition 1

Recall that a consensus implies that $\lim_{t\to\infty} W^t = \mathbf{e}\pi^T$. Let $\Pi = \mathbf{e}\pi^T$, and note that since Π derives from infinitely iterating W, then $\Pi W = \Pi$. It follows from the structure of Π as a matrix with identical rows of π that

$$\pi^T = \pi^T W \tag{A.2}$$

From (A.2), the vector π is the left eigenvector of a row-stochastic matrix W corresponding to the leading eigenvalue $\lambda_1 = 1$. To proof that the dynamics system (3) converges to a consensus, it suffices to show that there exists a unique vector π that solves the system of equation represented by (A.2).

We proceeds by first rewriting the interaction matrix W as a function of another matrix $\bar{W}$ as follows. Define $q_i \equiv \min_{1\leq j\leq n} w_{ji}$ for each $i \in N$, and let $\mathbf{q} = (q_1, q_2, \cdots, q_n)^T$. Each q_i is the least weight attached to i by other agents. Since the network consists of at least one central agent, that is there exists at least one agent i for whom $w_{ij} > 0$ for all $j \in N$, then $\mathbf{q}$ has at least one i with $q_i > 0$.

Now, let $\beta = \sum_{i\in N} q_i$ and define a vector $\bar{\mathbf{x}}$ as $\bar{\mathbf{x}} = \frac{1}{\beta}\mathbf{q}$. The vector $\bar{\mathbf{x}}$ is thus a normalized version of $\mathbf{q}$ so that $\sum_{i\in N} \bar{x}_i = 1$. All these definitions and properties are unique to networks with at least one central agent. We then define another matrix $\bar{W}$ as

$$\bar{W} = \frac{1}{1-\beta}\left(W - \beta\mathbf{e}\bar{\mathbf{x}}^T\right) \tag{A.3}$$

where $\mathbf{e}\bar{\mathbf{x}}^T$ is a matrix with identical rows of $\bar{\mathbf{x}}$. Consequently, each element of $\bar{W}$ is $\bar{w}_{ij} = \frac{1}{1-\beta}\left(w_{ij} - x_j\right)$, and W can be equivalently expressed as.

$$W = (1-\beta)\bar{W} + \beta\mathbf{e}\bar{\mathbf{x}}^T \tag{A.4}$$

Note that by definition, if at least one central agent exists, then $\beta \in (0, 1]$. Note also that just like W, $\bar{W}$ is also row-stochastic. That is

$$\bar{W}\mathbf{e} = \frac{1}{1-\beta}\left(W\mathbf{e} - \beta\mathbf{e}\bar{\mathbf{x}}^T\mathbf{e}\right) = \frac{1}{1-\beta}\left(\mathbf{e} - \beta\mathbf{e}\right) = \frac{1}{1-\beta}\left(1-\beta\right)\mathbf{e} = \mathbf{e}$$

To find π, we then substitute W as defined in (A.4) into (A.2) to yield

$$\pi^T W = (1-\beta)\pi^T\bar{W} + \beta\pi^T\mathbf{e}\bar{\mathbf{x}}^T = (1-\beta)\pi^T\bar{W} + \beta\bar{\mathbf{x}}^T$$

And hence, letting I denote an identity matrix, π is then given by

$$\pi^T = \beta\bar{\mathbf{x}}^T\left[I - (1-\beta)\bar{W}\right]^{-1} = \beta\bar{\mathbf{x}}^T\left[\sum_{\tau=0}^{\infty}(1-\beta)^\tau\bar{W}^\tau\right] = \beta\sum_{\tau=0}^{\infty}\bar{\mathbf{x}}^T\left[(1-\beta)^\tau\bar{W}^\tau\right] \tag{A.5}$$

We know from Debreu and Herstein (1953, Theorem III) that is $\bar{W}$ is indecomposable, which is true in our case since there exists at least one central agent, then $\left[I - (1-\beta)\bar{W}\right]^{-1} > 0$, that is a matrix with positive entries, if and only if $\lambda_1(\bar{W}) \leq \frac{1}{1-\beta}$, where $\lambda_1(\bar{W})$ is the largest eigenvalue of $\bar{W}$. Since $\bar{W}$ is row-stochastic, $\lambda_1(\bar{W}) = 1$; hence $\left[I - (1-\beta)\bar{W}\right]^{-1} > 0$ if and only if $\frac{1}{1-\beta} > 1$, which holds since $\beta \in (0, 1]$. It then follows from (A.5) that the vector π is unique and non-zero.

Appendix A.3. Proof of Proposition 2

Let $N_j(-i) = N_j \backslash N_i$ be the set of agents in the neighbourhood of j excluding all agents belonging to the neighbourhood of i; Just N_j, $N_j(-i)$ also excludes j. Let $w_j(-i) = \sum_{k \in N_j(-i)} w_{kj}$ be the total weight that agents in $N_j(-i)$ attach to j. Then from (5), π_j can be re-written as

$$\pi_j = \sum_{k \in N_j(-i)} w_{kj}\pi_k + \sum_{l \in N_j(i)} w_{lj}\pi_l \tag{A.6}$$

The difference $\pi_i - \pi_j$ is then

$$\begin{aligned}
\pi_i - \pi_j &= \sum_{k \in N_i} w_{ki}\pi_k - \sum_{k \in N_j(i)} w_{kj}\pi_k - \sum_{l \in N_j(-i)} w_{lj}\pi_l \\
&= \sum_{k \in N_i} (w_{ki} - w_{kj})\pi_k - \sum_{l \in N_j(-i)} w_{lj}\pi_l && j \text{ is a central agent: } N_j(i) = N_i \\
&\leq \sum_{k \in N_i} (w_{ki} - w_{kj}) - \sum_{l \in N_j(-i)} w_{lj}\pi_l && \text{assumes that } \sum_{k \in N_i} w_{ki} > \sum_{k \in N_i} w_{kj} \\
&= (w_i - w_j(i)) - \sum_{l \in N_j(-i)} w_{lj}\pi_l && \text{(A.7)}
\end{aligned}$$

Hence, given $w_j(-i)$ and $\sum_{l \in N_j(-i)} w_{lj}\pi_l$, we can choose $w_i > w_j(i)$ so that $\pi_i > \pi_j$.

Appendix A.4. Proof of Proposition 3

The proof proceeds by first deriving the vector of influences $\pi(t)$ at t, then deriving bounds for the supremum of $DC(t; W; \mathbf{a}(0))$, which we then use to derive bounds for the convergence time. Recall the expression for equilibrium influence vector from Section Appendix A.2,

$$\pi^T = \beta\bar{\mathbf{x}}^T \left[I - (1-\beta)\bar{W}\right]^{-1} = \beta\bar{\mathbf{x}}^T \left[\sum_{\tau=0}^{\infty} (1-\beta)^\tau \bar{W}^\tau\right] = \beta \sum_{\tau=0}^{\infty} \bar{\mathbf{x}}^T \left[(1-\beta)^\tau \bar{W}^\tau\right] \tag{A.8}$$

where $\bar{\mathbf{x}} = \frac{1}{\beta}\mathbf{q}$, and $\mathbf{q}$ and β are as defined above; $\bar{\mathbf{x}}$ is thus a normalized version of $\mathbf{q}$ so that $\sum_{i \in N} \bar{x}_i = 1$, and where the relationship between W and $\bar{W}$ is

$$W = (1-\beta)\bar{W} + \beta\mathbf{e}\bar{\mathbf{x}}^T \tag{A.9}$$

As shown in Section Appendix A.2, $\bar{W}$ is non-negative and row stochastic just as W. Define matrices $M^{[\beta]}(t)$ and $M^{[\beta]}(\infty)$ respectively as follows

$$M^{[\beta]}(t) = \frac{\beta}{1-(1-\beta)^{t+1}} \sum_{\tau=0}^{t} \left[(1-\beta)^\tau \bar{W}^\tau\right]$$

$$M^{[\beta]}(\infty) = \beta \sum_{\tau=0}^{\infty} \left[(1-\beta)^\tau \bar{W}^\tau\right]$$

From (A.8), it follows that $\pi^T = \bar{\mathbf{x}}^T M^{[\beta]}(\infty)$. For a matrix $\bar{W}$ and for each pair of agents i and j, $m_{ij}^{[\beta]}(\infty)$, the element in the ith-row and jth-column of $M^{[\beta]}(\infty)$, is the level of influence that

j exerts on i's opinions in equilibrium. Put differently, it is the expected normalized number of times j's opinion is incorporated in i's opinion. A similar interpreation follows for $m_{ij}^{[\beta]}(t)$ of $M^{[\beta]}(t)$ after t iterations. This interpretation is supported by the following claim.

Claim 1. *For a random walk process on $(1-\beta)\bar{W}$, $\frac{1-(1-\beta)^{t+1}}{\beta}\sum_{\tau=1}^{t}((1-\beta)\bar{w}_{ij})^{\tau}$ is the expected normalized number of visits to j starting from i after t iterations*

Claim 1 follows from the following argument. Let Y_t be the random walk process on the matrix $(1-\beta)\bar{W}$, and let $I_t = 1$ if the process is in j at period t and zero otherwise, so that the number of visits to j from i in t transitions is $\sum_{\tau}^{t} I_\tau$. Viewed this way, the quantity $1-\beta$ is the probability that at any given time t, the random walk process occurs; and if it occurs, if follows the transition probabilities described by $\bar{W}$. Let $P(E)$ be the probability of event E. The expected number of visits to j starting from i after t transitions is then

$$\begin{aligned}\mathbb{E}\left[\sum_{\tau}^{t} I_\tau \mid Y_0 = i\right] &= \sum_{\tau}^{t}\mathbb{E}\Big[I_\tau \mid Y_0 = i\Big] = \sum_{\tau}^{t}\Big[1\times P(I_\tau \mid Y_0 = i) + 0\times(1-P(I_\tau \mid Y_0 = i))\Big] \\ &= \sum_{\tau}^{t} P(I_\tau \mid Y_0 = i) = \sum_{\tau=1}^{t}((1-\beta)\bar{w}_{ij})^{\tau}\end{aligned}$$

Normalizing by a factor of $\sum_{\tau=0}^{t}(1-\beta)^{\tau} = \frac{1-(1-\beta)^{t+1}}{\beta}$, which is the total number of times the random walk process follows the transition probabilities described by $\bar{W}$, then implies that $m_{ij}^{[\beta]}(t) = \frac{\beta}{1-(1-\beta)^{t+1}}\sum_{\tau=0}^{t}((1-\beta)\bar{w}_{ij})^{\tau}$ is the expected normalized number of visits to j after t iterations starting from i. In relation to above definitions, this value is then the level of influence that j exerts on i. And hence, it is the normalized number of times j's opinion is incorporated into i's opinion after t iterations. In the long-run,

$$\lim_{t\to\infty} M^{[\beta]}(t) = \lim_{t\to\infty}\frac{\beta}{1-(1-\beta)^{t+1}}\left[\sum_{\tau=0}^{t}((1-\beta)\bar{W})^{\tau}\right] = \beta\sum_{\tau=0}^{\infty}((1-\beta)\bar{W})^{\tau} = M^{[\beta]}(\infty)$$

where the second equality follows from $\lim_{t\to\infty}\frac{\beta}{1-(1-\beta)^{t+1}} = \beta$. Given $M^{[\beta]}(t)$, the influence vector $\pi(t)$ after t iterations is then $\pi^T(t) = \bar{\mathbf{x}}^T M^{[\beta]}(t)$. We would then like to establish bounds for $DC(t;W) = \sup_{\mathbf{a}(0)\in\mathbb{R}^n} DC(t;W;\mathbf{a}(0))$, where

$$DC(t;W;\mathbf{a}(0)) = \sum_{i=1}^{n}|\pi_i(t)a_i(0) - \pi_i a_i(0)| \tag{A.10}$$

First note that

$$DC(t;W;\mathbf{a}(0)) = \sum_{i=1}^{n}|\pi_i(t)a_i(0) - \pi_i a_i(0)| = \sum_{i=1}^{n}|(\pi_i(t)-\pi_i)\,a_i(0)| = \sum_{i=1}^{n}|\pi_i(t)-\pi_i|\,|a_i(0)| \tag{A.11}$$

But since $a_i(0)\in(0,1]$, then the supremum of $DC(t;W;\mathbf{a}(0))$ is

$$DC(t;W) = \sup_{\mathbf{a}(0)\in\mathbb{R}^n}\sum_{i=1}^{n}|\pi_i(t)-\pi_i|\,|a_i(0)| = \sum_{i=1}^{n}|\pi_i(t)-\pi_i| \tag{A.12}$$

The second equality on the right hand side of (A.12) is equivalent to the Euclidean distance between vectors $\pi(t)$ and π; that is

$$DC(t;W) = \sum_{i=1}^{n} |\pi_i(t) - \pi_i| = \|\pi(t) - \pi\|_1 \tag{A.13}$$

Substituting the expressions for $\pi(t)$ and π into (A.13) yields

$$\begin{aligned} DC(t;W) &= \left\|\bar{\mathbf{x}}^T \bar{M}^{[\beta]}(t) - \pi^T\right\|_1 \\ &= \beta \left\| \frac{1}{1-(1-\beta)^{t+1}} \sum_{\tau=0}^{t} (1-\beta)^\tau \left[\bar{\mathbf{x}}^T \bar{W}^\tau\right] - \sum_{\tau=0}^{\infty} (1-\beta)^\tau \left[\bar{\mathbf{x}}^T \bar{W}^\tau\right] \right\|_1 \end{aligned} \tag{A.14}$$

Let $\bar{\mathbf{x}}^T(\tau) = \bar{\mathbf{x}}^T \bar{W}^\tau$, so that $\{\bar{\mathbf{x}}(\tau)\}_{\tau \geq 0}$ is a random walk on $\bar{W}$ with initial vector $\bar{\mathbf{x}}$. This interpretation of $\mathbf{x}(t)$ is feasible since $\bar{W}$ is a row-stochastic matrix. It follows from the property of random walks that $\|\bar{\mathbf{x}}(t)\|_1 = \sum_{i=1}^{n} \mid \bar{x}_i(t) \mid = 1$ at each $t = 0, 1, \cdots$ From (A.14), we have

$$\begin{aligned} DC(t;W) &= \beta \left\| \frac{1}{1-(1-\beta)^{t+1}} \sum_{\tau=0}^{t} (1-\beta)^\tau \bar{\mathbf{x}}(\tau) - \sum_{\tau=0}^{\infty} (1-\beta)^\tau \bar{\mathbf{x}}(\tau) \right\|_1 \\ &= \beta \left\| \frac{(1-\beta)^{t+1}}{1-(1-\beta)^{t+1}} \sum_{\tau=0}^{t} (1-\beta)^\tau \bar{\mathbf{x}}(\tau) - \sum_{\tau=t+1}^{\infty} (1-\beta)^\tau \bar{\mathbf{x}}(\tau) \right\|_1 \end{aligned} \tag{A.15}$$

Using triangular inequality gives

$$\begin{aligned} DC(t;W) &\leq \beta \left\| \frac{(1-\beta)^{t+1}}{1-(1-\beta)^{t+1}} \sum_{\tau=0}^{t} (1-\beta)^\tau \bar{\mathbf{x}}(\tau) \right\|_1 + \left\| \sum_{\tau=t+1}^{\infty} (1-\beta)^\tau \bar{\mathbf{x}}(\tau) \right\|_1 \\ &\leq \beta \left[\frac{(1-\beta)^{t+1}}{1-(1-\beta)^{t+1}} \sum_{\tau=0}^{t} (1-\beta)^\tau \|\bar{\mathbf{x}}(\tau)\|_1 + \sum_{\tau=t+1}^{\infty} (1-\beta)^\tau \|\bar{\mathbf{x}}(\tau)\|_1 \right] \\ &= \beta \left[\frac{(1-\beta)^{t+1}}{1-(1-\beta)^{t+1}} . \frac{1-(1-\beta)^{t+1}}{\beta} + \frac{(1-\beta)^{t+1}}{\beta} \right] \\ &= 2\,(1-\beta)^{t+1} \end{aligned} \tag{A.16}$$

For the lower bound, we use the following results on reverse triangular inequality from Diaz and Metcalf (1966, Theorem 1). Let $\mathbf{u}$ be a unit vector in the Hilbert Space. Suppose the vectors $\bar{\mathbf{x}}(1), \cdots, \bar{\mathbf{x}}(n)$, whenever $\bar{\mathbf{x}}(\tau) \neq 0$ satisfies $0 \leq r \leq \frac{Re\langle \bar{\mathbf{x}}(\tau), \mathbf{u}\rangle}{\|\bar{\mathbf{x}}(\tau)\|}$, where Re stands for "real number" and $\langle \mathbf{x}, \mathbf{y}\rangle = \sum_{i=1}^{n} x_i y_i$, is the inner product of $\mathbf{x}$ and $\mathbf{y}$. Then

$$\left\| \sum_{i=1}^{n} \bar{\mathbf{x}}(\tau) \right\| \geq r \sum_{i=1}^{n} \|\bar{\mathbf{x}}(\tau)\| \tag{A.17}$$

Applying inequality (A.17) to (A.14) yields

$$
\begin{aligned}
DC(t;W) &= \beta \left\| \frac{(1-\beta)^{t+1}}{1-(1-\beta)^{t+1}} \sum_{\tau=0}^{t} (1-\beta)^{\tau} \bar{\mathbf{x}}(\tau) - \sum_{\tau=t+1}^{\infty} (1-\beta)^{\tau} \bar{\mathbf{x}}(\tau) \right\|_1 \\
&\geq r\beta \left[\frac{(1-\beta)^{t+1}}{1-(1-\beta)^{t+1}} \sum_{\tau=0}^{t} (1-\beta)^{\tau} \left\| \bar{\mathbf{x}}(\tau) \right\|_1 + \sum_{\tau=t+1}^{\infty} (1-\beta)^{\tau} \left\| \bar{\mathbf{x}}(\tau) \right\|_1 \right] \\
&= r\beta \left[\frac{(1-\beta)^{t+1}}{1-(1-\beta)^{t+1}} . \frac{1-(1-\beta)^{t+1}}{\beta} + \frac{(1-\beta)^{t+1}}{\beta} \right] \\
&= 2r\,(1-\beta)^{t+1} \qquad \text{(A.18)}
\end{aligned}
$$

We now derive the bounds for r based on the definition above. The summations on the right hand side of (A.15) consists of vectors $\bar{\mathbf{x}}(\tau)$ for $\tau = 0, 1, 2, \cdots$. To derive an upper bound for $\frac{Re\langle \bar{\mathbf{x}}(\tau), \mathbf{u} \rangle}{\|\bar{\mathbf{x}}(\tau)\|}$, we choose a τ so that the corresponding $\bar{\mathbf{x}}(\tau)$ maximizes the former. Since the distance to consensus, $DC(t;W)$, decays over time, the largest vector in the summation of the first equality of (A.15) is when $\tau = 0$; denoting it by $\mathbf{v}$, we have

$$
\mathbf{v} = \beta \frac{1}{1-(1-\beta)} (1-\beta)^0 \bar{\mathbf{x}}(0) - \beta (1-\beta)^0 \bar{\mathbf{x}}(0) = (1-\beta)\bar{\mathbf{x}}
$$

Let $\|\bar{\mathbf{x}}\|_2 = \left(\sum_{i=1}^{n} \bar{x}_i^2\right)^{\frac{1}{2}}$ be the length of vector $\bar{\mathbf{x}}$. We can then define an arbitrary unite vector $\mathbf{u} = \frac{1}{\|\bar{\mathbf{x}}\|_2} \bar{\mathbf{x}}$. Note also that $\|\mathbf{v}\| = (1-\beta)\,\|\bar{\mathbf{x}}\| = (1-\beta)$. Substituting into $\frac{Re\langle \bar{\mathbf{x}}(\tau), \mathbf{u} \rangle}{\|\bar{\mathbf{x}}(\tau)\|}$ yields

$$
\frac{Re\langle \bar{\mathbf{x}}(\tau), \mathbf{u} \rangle}{\|\bar{\mathbf{x}}(\tau)\|} \leq \frac{Re\langle \mathbf{v}, \mathbf{u} \rangle}{\|\mathbf{v}\|} = \left(\frac{1-\beta}{\|\bar{\mathbf{x}}\|_2} \right) \frac{\langle \bar{\mathbf{x}}, \bar{\mathbf{x}} \rangle}{1-\beta} = \frac{1}{\|\bar{\mathbf{x}}\|_2} \langle \bar{\mathbf{x}}, \bar{\mathbf{x}} \rangle = \frac{1}{\|\bar{\mathbf{x}}\|_2} \|\bar{\mathbf{x}}\|_2^2 = \|\bar{\mathbf{x}}\|_2
$$

And hence $0 \leq r \leq \left(\sum_{i=1}^{n} \bar{x}_i^2\right)^{\frac{1}{2}} \leq \left(\left(\sum_{i=1}^{n} \bar{x}_i\right)^2\right)^{\frac{1}{2}} = 1$. Since $\bar{\mathbf{x}} = \frac{1}{\beta}\mathbf{q}$, it follows that $\left(\sum_{i=1}^{n} \bar{x}_i^2\right)^{\frac{1}{2}} = \frac{1}{\beta}\left(\sum_{i=1}^{n} q_i^2\right)^{\frac{1}{2}}$.

Since $\beta \neq 0$, that is there is at least one central agent in the society, then from the upper bound of $DC(t;W)$ in (A.16), if $t \geq \frac{\ln(\varepsilon/2)}{\ln\left(1-\beta\right)} - 1$, then $DC(t;W) \leq \varepsilon$. The upper bound for $CT(\varepsilon;W)$ then follows from Definition 1.

Similarly from (A.18), if $t \leq \frac{\ln(\varepsilon/2r)}{\ln\left(1-\beta\right)} - 1$, then $DC(t;W) \geq \varepsilon$. The lower bound for $CT(\varepsilon;W)$ also then follows from its definition 1.

Appendix A.5. Proof of Corollary 1

Recall the following definitions of W and $\bar{W}$

$$
W = (1-\beta)\bar{W} + \beta \mathbf{e}\bar{\mathbf{x}}^T \qquad \text{(A.19)}
$$

$$
\bar{W} = \frac{1}{1-\beta}\left(W - \beta \mathbf{e}\bar{\mathbf{x}}^T\right) \qquad \text{(A.20)}
$$

Recall also that $\beta = \sum_{i \in N} q_i$ and $\bar{\mathbf{x}} = \frac{1}{\beta}\mathbf{q}$. We can thus rewrite (A.20) as

$$\bar{W} = \frac{1}{1-\beta}\left(W - \mathbf{e}\mathbf{q}^T\right) \tag{A.21}$$

Define another matrix

$$\hat{W} = W - \mathbf{e}\mathbf{q}^T = (1-\beta)\bar{W} \tag{A.22}$$

and note that both W and $\bar{W}$ are row-stochastic; that is $W\mathbf{e} = \mathbf{e}$ and $\bar{W}\mathbf{e} = \mathbf{e}$. Let $\mathbf{v}_j$ be the left eigenvector of W corresponding to a non-negative eigenvalue μ_j. Then $\mathbf{v}_j^T\mathbf{e} = \mathbf{v}_j^T W\mathbf{e} = \mu_j\mathbf{v}_j^T\mathbf{e}$, which in turn implies that $\mathbf{v}_j^T\mathbf{e}(1-\mu_j) = 0$; and since $(1-\mu_j) \neq 0$, then $\mathbf{v}_j^T\mathbf{e} = 0$. The direct implication of the latter equality is that $\mathbf{v}_j^T\hat{W} = \mathbf{v}_j^T W = \mu_j\mathbf{v}_j^T$, so that $\hat{W}$ and W have similar non-zero eigenvalues and corresponding eigenvectors.

The result of proposition then follows from noting that $\hat{W}$ is non-negative, W and $\bar{W}$ are row-stochastic and hence $\mu_1 = 1$, and from (A.22)

$$\hat{W}\mathbf{e} = (1-\beta)\bar{W}\mathbf{e} = (1-\beta)\mathbf{e} \tag{A.23}$$

Hence, $\mu_1(\hat{W}) = (1-\beta)$ is the leading eigenvalue of $\hat{W}$ and since $\mathbf{v}_j^T\hat{W} = \mathbf{v}_j^T W = \mu_j\mathbf{v}_j^T$ for non-zero eigenvalues of W, it follows $\mu_2(W) = \mu_2(\hat{W}) \leq 1-\beta$.

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